

Differentiate Almost Everywhere

Differentiable Relaxations and Reparameterisations

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- But, those blocks need to be differentiable to work in our optimisation framework
 - More specifically they need to be continuous and *differentiable almost everywhere*.
- That limits what we can do... Can we work around that?
 - Relaxations — make continuous (and potentially differentiable everywhere) approximations.
 - Reparameterisations — rewrite functions to factor out stochastic variables from the parameters.

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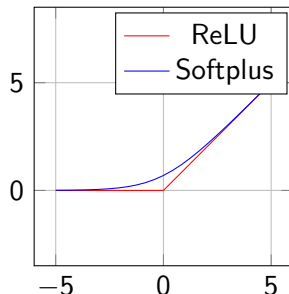
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 - There are *subgradients* at $x = 0$; implementations usually just arbitrarily pick $f'(0) = 0$
- Functions that are differentiable almost everywhere or have subgradients tend to be compatible with gradient descent methods
 - We expect that the loss landscape is different for each batch & that we'll never actually reach a minima, and we only need to *mostly* take steps in the right direction.

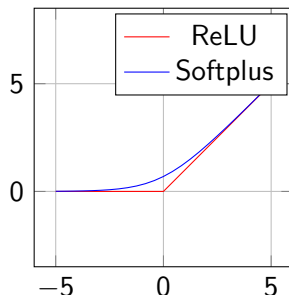
Relaxing ReLU

- Softplus ($\text{softplus}(x) = \ln(1 + e^x)$) is a relaxation of ReLU that is *differentiable everywhere*.



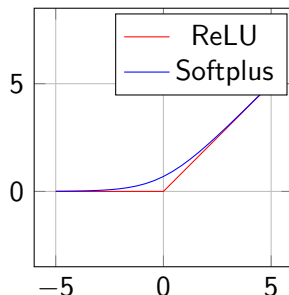
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- Softplus ($\text{softplus}(x) = \ln(1 + e^x)$) is a relaxation of ReLU that is *differentiable everywhere*.
- Its derivative is the Sigmoid function
- Not widely used; counter-intuitively, even though it neither saturates completely and is differentiable everywhere, empirically it has been shown that ReLU works better.



Interpretations of softmax

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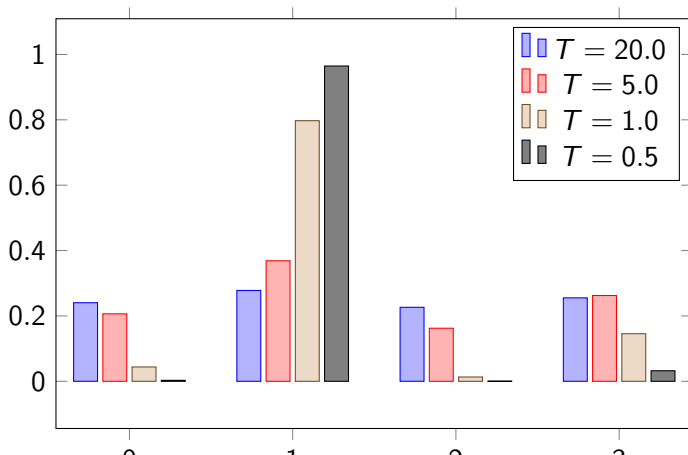
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 - but not of the max function like the name would suggest!
 - softmax can be viewed as a continuous and differentiable relaxation of the arg max function with one-hot output encoding.
 - The arg max function is not continuous or differentiable; softmax provides an approximation:

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 1.1 & 4.0 & -0.1 & 2.3 \end{bmatrix} \\ \arg \max(\mathbf{x}) &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \\ \text{softmax}(\mathbf{x}) &= \begin{bmatrix} 0.044 & 0.797 & 0.013 & 0.146 \end{bmatrix} \end{aligned}$$

The Softmax function with temperature

Consider what happens if you were to divide the input logits to a softmax by a scalar temperature parameter T .

$$\text{softmax}(\mathbf{x}/T)_i = \frac{e^{x_i/T}}{\sum_{j=1}^K e^{x_j/T}} \quad \forall i = 1, 2, \dots, K$$



arg max — softmax with temperature

$\mathbf{x} =$	[1.1	4.0	-0.1	2.3]
$\text{softmax}(\mathbf{x}/1.0) =$	[0.044	0.797	0.013	0.146]
$\text{softmax}(\mathbf{x}/0.8) =$	[0.023	0.868	0.005	0.104]
$\text{softmax}(\mathbf{x}/0.6) =$	[0.008	0.937	0.001	0.055]
$\text{softmax}(\mathbf{x}/0.4) =$	[6.997e-04	9.852e-01	3.484e-05	1.405e-02]
$\text{softmax}(\mathbf{x}/0.2) =$	[5.042e-07	9.998e-01	1.250e-09	2.034e-04]

arg max — scalar approximation

- What if you want to get a scalar approximation to the index of the arg max rather than a probability distribution approximating the one-hot form?
 - Caveat: we are not actually going to get a guaranteed integer representation as that would be non-differentiable; we'll have to live with a float that is an approximation¹.

¹for now — we'll address this in a few slides time!

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- First, consider how to convert a one-hot vector to index representation in a differentiable manner: $[0, 0, 1, 0] \rightarrow 2$
 - Just dot product with a vector of indices: $[0, 1, 2, 3]$

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- First, consider how to convert a one-hot vector to index representation in a differentiable manner: $[0, 0, 1, 0] \rightarrow 2$
 - Just dot product with a vector of indices: $[0, 1, 2, 3]$
- The same process can be applied to the softmax distribution
 - As temperature $T \rightarrow 0$, $\text{softmax}(\mathbf{x}/T) \cdot [0, 1, \dots, N] \rightarrow \arg \max(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^N$.

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arg max — scalar approximation

$$\mathbf{x} = [1.1 \quad 4.0 \quad -0.1 \quad 2.3]^\top$$

$$\mathbf{i} = [0.0 \quad 1.0 \quad 2.0 \quad 3.0]^\top$$

$$\text{softmax}(\mathbf{x}/1.0)^\top \mathbf{i} = 1.2606$$

$$\text{softmax}(\mathbf{x}/0.8)^\top \mathbf{i} = 1.1894$$

$$\text{softmax}(\mathbf{x}/0.6)^\top \mathbf{i} = 1.1037$$

$$\text{softmax}(\mathbf{x}/0.4)^\top \mathbf{i} = 1.0274$$

$$\text{softmax}(\mathbf{x}/0.2)^\top \mathbf{i} = 1.0004$$

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$$\mathbf{x} = [1.1 \quad 4.0 \quad -0.1 \quad 2.3]^\top$$

$$\text{softmax}(\mathbf{x}/1.0)^\top \mathbf{x} = 3.571$$

$$\text{softmax}(\mathbf{x}/0.8)^\top \mathbf{x} = 3.736$$

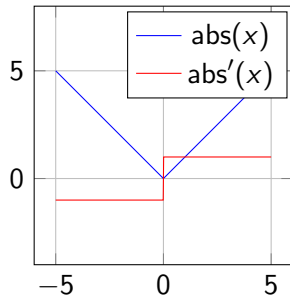
$$\text{softmax}(\mathbf{x}/0.6)^\top \mathbf{x} = 3.881$$

$$\text{softmax}(\mathbf{x}/0.4)^\top \mathbf{x} = 3.974$$

$$\text{softmax}(\mathbf{x}/0.2)^\top \mathbf{x} = 3.999$$

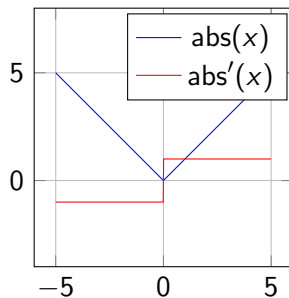
L1 norm

- L1 norm is the sum of absolute values of a vector



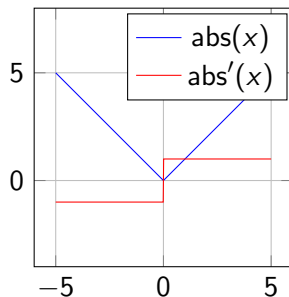
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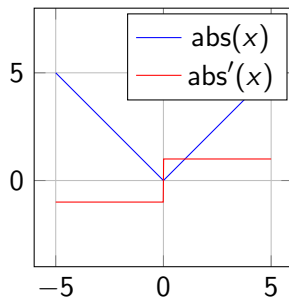
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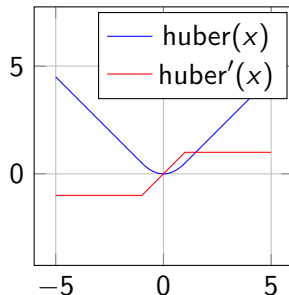
- L1 norm is the sum of absolute values of a vector
- We've seen that an L1 norm regulariser can induce sparsity in a model
- abs is continuous and differentiable almost everywhere, but...
- unlike ReLU, the gradients left and right of the discontinuity point in equal and opposite directions
 - This can cause oscillations that prevent or hamper learning



Relaxing the L1 norm

- Huber loss (aka Smooth L1 loss) relaxes L1 by mixing it with L2 near the origin:

$$z_i = \begin{cases} 0.5(x_i - y_i)^2, & \text{if } |x_i - y_i| < 1 \\ |x_i - y_i| - 0.5, & \text{otherwise} \end{cases}$$

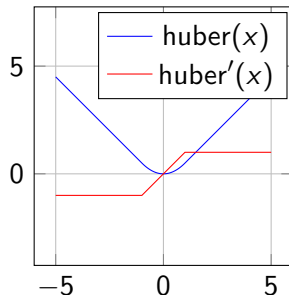


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- In both cases gradients reduce in magnitude and switch direction smoothly which can lead to much less oscillation.



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 - provided that f is continuous and differentiable (almost everywhere) we can perform gradient based optimisation as usual.

Consider

$$y \sim \mathcal{N}(\mu, \sigma^2)$$

How can we take derivatives of y with respect to μ and σ^2 ?

If we rewrite

$$y = \mu + \sigma z \text{ where } z \sim \mathcal{N}(0, 1)$$

Then it is clear that y is a function of a deterministic operation with variables μ and σ with an (extra) input z .

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- Crucially the extra input is an r.v. whose distribution is not a function of any variables whose derivatives we wish to calculate.
- The derivatives $dy/d\mu$ and $dy/d\sigma$ tell us how an infinitesimal change in μ or σ would change y if we could repeat the sampling operation with the *same* value of z

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 - A sample $y \sim p(y; \omega)$ can be rewritten as $y = f(z, \omega)$ where z is a source of randomness.
 - We can thus compute derivatives $\partial y / \partial \omega$ and use gradient based optimisation as long as
 - f is continuous and differentiable almost everywhere
 - ω is not a function of z
 - and z is not a function of ω

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- Potential solutions:
 - Policy Gradient Methods (e.g. the REINFORCE algorithm)
 - A relaxation and another 'trick': Gumbel Softmax and the Straight-through operator

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 - But, it can be estimated without bias using an Monte Carlo average.
- REINFORCE is a family of algorithms that utilise this idea.

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The simplest form of REINFORCE is easy to derive by differentiating the expected loss:

$$\mathbb{E}_{\mathbf{z}}[\mathcal{L}(\mathbf{y})] = \sum_{\mathbf{y}} \mathcal{L}(\mathbf{y}) p(\mathbf{y}) \quad (1)$$

$$\frac{\partial \mathbb{E}[\mathcal{L}(\mathbf{y})]}{\partial \boldsymbol{\omega}} = \sum_{\mathbf{y}} \mathcal{L}(\mathbf{y}) \frac{\partial p(\mathbf{y})}{\partial \boldsymbol{\omega}} \quad (2)$$

$$= \sum_{\mathbf{y}} \mathcal{L}(\mathbf{y}) p(\mathbf{y}) \frac{\partial \log p(\mathbf{y})}{\partial \boldsymbol{\omega}} \quad (3)$$

$$\approx \frac{1}{m} \sum_{\mathbf{y}^{(i)} \sim p(\mathbf{y}), i=1}^m \mathcal{L}(\mathbf{y}^{(i)}) \frac{\partial \log p(\mathbf{y}^{(i)})}{\partial \boldsymbol{\omega}} \quad (4)$$

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- This gives us an unbiased MC estimator of the gradient.
- Unfortunately this is a very high variance estimator, so it would require many samples of \mathbf{y} to be drawn to obtain a good estimate
 - or equivalently, if only one sample were drawn, SGD would converge very slowly and **require** a small learning rate.

Sampling from a categorical distribution: Gumbel Softmax

The generation of a discrete token, t , from a vocabulary of K tokens is achieved by sampling a categorical distribution

$$t \sim \text{Cat}(p_1, \dots, p_K); \sum_i p_i = 1.$$

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The gumbel-softmax reparameterisation allows us to sample directly using the logits:

$$t = \underset{i \in \{1, \dots, K\}}{\operatorname{argmax}} x_i + z_i$$

where z_1, \dots, z_K are i.i.d Gumbel(0,1) variates which can be computed from Uniform variates through $-\log(-\log(\mathcal{U}(0,1)))$.

Differentiable Sampling: Straight-Through Gumbel Softmax

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...but we've already seen that we can relax arg max using

$$\text{softargmax}(\mathbf{y}) = \sum_i \frac{e^{y_i/T}}{\sum_j e^{y_j/T}} i$$

where T is the temperature parameter.

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The Straight-Through operator allows us to take the result of a true argmax that has the gradient of the softargmax :

$$\text{STargmax}(\mathbf{y}) = \text{softargmax}(\mathbf{y}) + \text{stopgradient}(\text{argmax}(\mathbf{y}) - \text{softargmax}(\mathbf{y}))$$

where stopgradient is defined such that $\text{stopgradient}(\mathbf{a}) = \mathbf{a}$ and $\nabla \text{stopgradient}(\mathbf{a}) = 0$.

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The Straight-Through operator allows us to take the result of a true argmax that has the gradient of the softargmax :

$$\text{STargmax}(\mathbf{y}) = \text{softargmax}(\mathbf{y}) + \text{stopgradient}(\text{argmax}(\mathbf{y}) - \text{softargmax}(\mathbf{y}))$$

where stopgradient is defined such that $\text{stopgradient}(\mathbf{a}) = \mathbf{a}$ and $\nabla \text{stopgradient}(\mathbf{a}) = 0$.

Straight-Through Gumbel Softmax

Combine the gumbel softmax trick with the STargmax to give you discrete samples, with a usable gradient^a.

^aThe ST operator is biased but low variance; in practice it works very well and is better than the high-variance unbiased estimates you could get through REINFORCE.

- Differentiable programming works with functions that are continuous and differentiable almost everywhere.
- Some non-continuous functions can be relaxed to make them more amenable to gradient based optimisation by making continuous approximations.
- Some continuous functions with discontinuous gradients can be relaxed to make optimisation more stable.
- Reparameterisations can allow us to differentiate through random operations such as sampling
- We can even make networks output/utilise discrete variables by combining relaxations and reparameterisations.