# Differentiate your Objective

VLC = ∭ Vision Learning Control dVdLdC

# Differentiable Programming

## How does pre-university calculus relate to AI and the future of computer programming?

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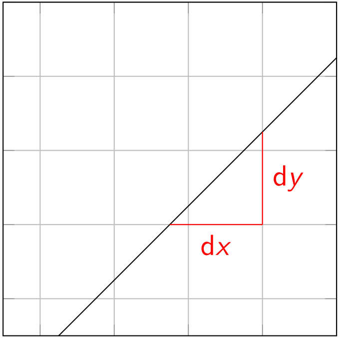
# Differentiation

# Recap: what is the derivative of a function of one variable?

## The derivative in 1D

- Recall that the gradient of a straight line is $\frac{\mathrm{d}y}{\mathrm{d}x}$.



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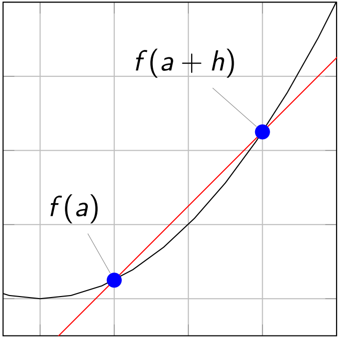
- For an arbitrary real-valued function, $f$, we can approximate the derivative at a point $a$, $f'(a)$, using the gradient of the *secant line* defined by $(a, f(a))$ and a point a small distance, $h$, away $(a+h, f(a+h))$: $f'(a) \approx \frac{f(a+h)-f(a)}{h}$.



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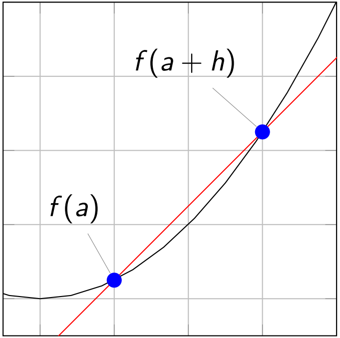

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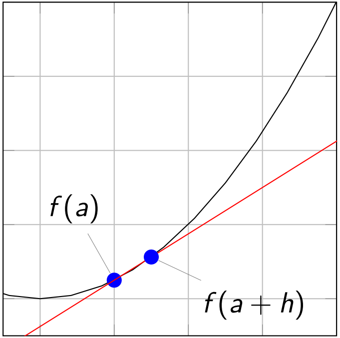

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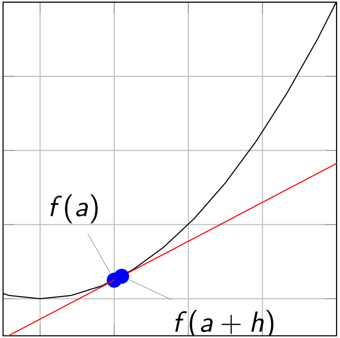
  $f'(a) \approx \frac{f(a+h)-f(a)}{h}$.
    - This expression is *Newton's Quotient*.
    - As $h$ becomes smaller, the approximated derivative becomes more accurate.
    - If we take the limit as $h \to 0$, then we have an exact expression for the derivative:
      $\frac{df}{da} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.



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\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x
\end{aligned}
$$

# Intuition: What does the gradient $\mathrm{d}y/\mathrm{d}x$ tell us

- The 'rate of change' of $y$ with respect to $x$.

# Intuition: What does the gradient $dy/dx$ tell us

- The 'rate of change' of $y$ with respect to $x$.
- By how much does $y$ change if I make a small change to the $x$.

- At what angle should a javelin be thrown to maximise the distance travelled?

# Why should we care?

## Solving a simple problem with differentiation

- At what angle should a javelin be thrown to maximise the distance travelled?
- Assume initial velocity $u = 28\,\mathrm{m\,s^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$
- Choose to ignore launch height as it is negligable compared to distance travelled.

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- Choose to ignore launch height as it is negligable compared to distance travelled.
- Kinematics equations:

$$x = ut\cos(\theta) = 28t\cos(\theta)$$
$$y = ut\sin(\theta) - 0.5gt^2 = 28t\sin(\theta) - 4.9t^2$$

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$$x = 28t\cos(\theta)$$
$$y = 28t\sin(\theta) - 4.9t^2$$

- Javelin hits ground when $y = 0$ and we only care about $t > 0$:

$$0 = 28t\sin(\theta) - 4.9t^2$$
$$\implies t = \frac{28}{4.9}\sin(\theta)$$

- Substituting into the horizontal component:

$$x = 28\frac{28}{4.9}\sin(\theta)\cos(\theta) = 80\sin(2\theta)$$

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$$\begin{aligned} \max_{\theta} \quad & 80 \sin(2\theta) \\ \text{s.t.} \quad & 0 \leq \theta \leq \frac{\pi}{2} \end{aligned}$$

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0 &= \frac{\mathrm{d}(80\sin(2\theta))}{\mathrm{d}\theta} \\
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**Irrespective of the initial velocity maximum distance is acheived at** $45^{\circ}$.

# Abstraction: Solving problems by minimising an objective

- To compute the parameter (angle) for the javelin example we *maximised* the equation for distance travelled.

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- We can solve all kinds of problems if we can:
  - **formulate** a *loss* or *cost* function.
  - **minimise** the loss with respect to the parameter(s)[1].
- Problems:
  - The loss must be differentiable (or rather you must be able to compute or estimate its gradient somehow).
  - The loss function could be arbitrarily complex... you might not be able to analytically compute the solution (or the gradient).
  - Some loss functions might have many minima; you might have to settle for finding a sub-optimal one (or a saddle-point).

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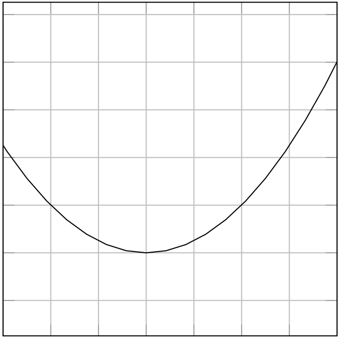
- How can you numerically estimate the value of the parameter $\theta$ that minimises a loss function, $\mathcal{L}(\theta)$?



# A simple algorithm for minimising a function

## Gradient Descent

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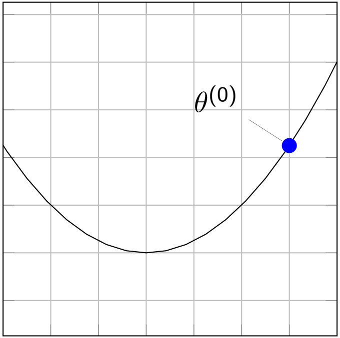
- Really intuitive idea: starting from an initial guess, $\theta^{(0)}$, take small steps in the direction of the negative gradient.



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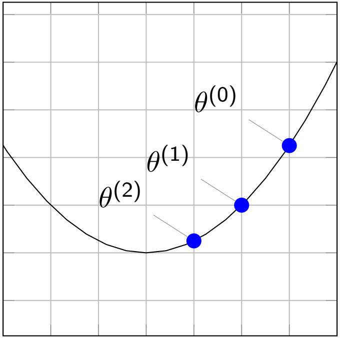
Gradient Descent:

$\theta^{(i+1)} = \theta^{(i)} - \lambda \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\theta}$ where $\lambda$ is the *learning rate*



# Javelin throwing again, but with Python code

# Derivatives of more general functions

- Almost all complex functions can be broken into simpler parts (often with very simple derivatives).
- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
    - The sum rule, product rule and chain rule tell you how to differentiate these.

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- Almost all complex functions can be broken into simpler parts (often with very simple derivatives).
- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
    - The sum rule, product rule and chain rule tell you how to differentiate these.
- If you break down functions into their constituent parts computing the derivative becomes very easy
- Example: the sin function can be written in terms of exponentials (Euler's formula) and the derivative of an exponential $e^x$ is just $e^x$...

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- Most interesting functions that we might want to work with have more than one parameter that we might want to optimise.
  - In many real applications it can be *millions* of parameters.
- Partial derivatives $\frac{\partial f}{\partial x_i}$ let us compute the gradient of the $i$-th parameter by holding the other parameters constant.

# Back to programming

# Programming is really just function composition and control statements

- At the end of the day computer programs are just compositions of really simple functions that computer processors can compute: arithmetic operations (add, multiply, divide, ...), logical operations (and, or, not, comparisons...), operations that move data, etc.

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- Many of these primitive operations have *well defined* gradients with respect to their operands.
- The chain rule tells us how to compute gradients of composite functions.

So, in principle we can find the optimal "parameters" of a computer program designed to solve a specific task by following the gradients to optimise it.

# Differentiating Branches

## Code - *if-else* statement

```
if a > 0.5:
    b = 0
else:
    b = 2 * a
```

## Math

$$b(a) = \begin{cases} 0 & \text{if } a > 0.5 \\ 2a & \text{if } a \leq 0.5 \end{cases}$$

$$\frac{\partial b}{\partial a} = \begin{cases} 0 & \text{if } a > 0.5 \\ 2 & \text{if } a \leq 0.5 \end{cases}$$

# Differentiating Loops

## Code - *for loop* statement

```
b = 1
for i in range(3):
    b = b + b * a
```

## Math

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= b_0 + b_0 a = 1 + a \\
b_2 &= b_1 + b_1 a = 1 + 2a + a^2 \\
b_3 &= b_2 + b_2 a = 1 + 3a + 3a^2 + a^3 \\
\frac{\partial b}{\partial a} &= 3 + 6a + 3a^2
\end{aligned}
$$

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- not every operation or function has *useful* gradients
  - discontinuities, large areas of zero-gradient, ...
- Computer science researchers are actively developing mathematical 'tricks' to circumvent many of these problems.
  - *Relaxations* of functions that behave almost the same, but have well defined gradients.
  - *Reparameterisations* of functions involving randomness.
  - *Approximations* of useable gradients for functions that have ill-posed gradients.

# What kinds of functional building blocks are common?

- Today, the most common operations with parameters are:
    - *Vector addition*: the input vector to a function is added to a vector of weights.
    - *Vector-Matrix multiplication*: the input vector to the function is multiplied with a matrix of weights.
    - *Convolution*: the input vector (or matrix...) is 'convolved' with a set of weights.
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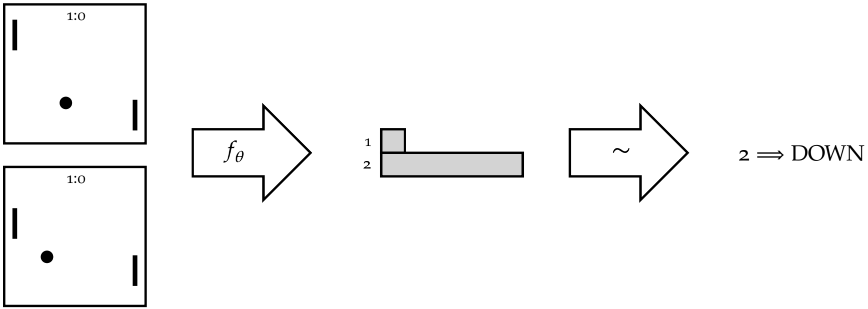
- The above operations are *linear*, so they are often combined with element-wise nonlinearities; e.g.:
  - $\max(0, x)$ aka ReLU.
  - $\tanh(x)$.
  - $\frac{1}{1+e^{-x}}$ aka *sigmoid* or the *logistic* function.

# Real Examples of Differentiable Programming

# Playing Games

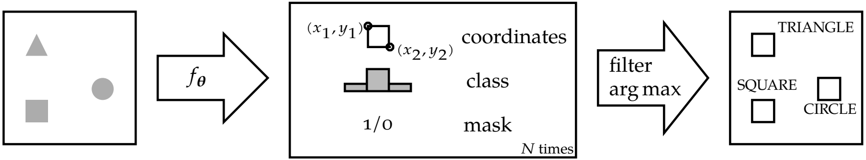
- You can use differentiable programming to write (and train) 'agents' that can play games.
- It can be hard to get a gradient from a single game involving many moves, but there is a clever trick which allows good estimates of gradients to be created over the average of *many* games.
- This is broadly the area of what is called *reinforcement learning*.



## Demo: AlphaStar

# Object detection

- Consider a function that takes an image as input and produces an array of *bounding boxes* and corresponding *labels*.
- With enough *training data* we can learn the parameters required to detect objects in images.



# Object detection

## Demo

# Drawing

- We could envisage a differentiable function that takes in a set of line coordinates and turns them into an image...
- With such a function we can optimise the line coordinates so they e.g. match a photograph, thus automatically creating a *sketch*.

# Drawing
## Demo

# Where is this all going?

# Software 2.0

There is a revolution happening and you're going to be part of it!

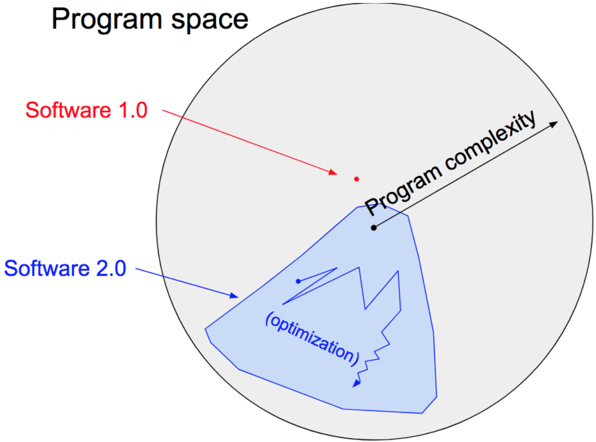



Image credit: Andrei Karpathy

https://karpathy.medium.com/software-2-0-a64152b37c35

## Any Questions?